ANALYSES OF THE SOLIDIFICATION OF CYLINDRICALLY SHAPED INGOTS
BY THE LEIBENZON METHOD

Yu. A. Samoilovich and V. I. Timoshpol'skii   UDC 536.24:621.747

An analytic solution is obtained of the problem about the dynamics of the solidi-
fication of alloys of circular transverse section which takes account of the pro-
gress of the two-phase zone boundaries in time.

A simplified method based on the assumption of the existence of a smooth phase interface
where the physical properties of the solidifying substance changes by a jump is used exten-
sively in the analysis of ingot solidification [1, 2]. Meanwhile, the majority of industrial
alloys crystallizes in the temperature range T_{liq}-T_{sol}. As is shown in Fig. 1, the presence
of a finite (nonzero) crystallization temperature band for a nonuniform temperature distribu-
tion over the ingot section will result in the existence of a two-phase zone (melt, crystals)
where the relative quantity of solid phase ($\psi = V_s/V_0$) changes from zero (for $T = T_{liq}$) to
one (for $T = T_{sol}$). Extension of the Ivantsov—Veinik method [1, 2] to the case of ingot
crystallization in the T_{liq}-T_{sol} temperature band is performed in [3] for the case of solidi-
fication of plane ingots for a boundary condition of the first kind (T_{sur} = const) on their
cooling surface. A solution is presented below for an analogous problem for ingots of cylin-
drical shape with circular cross section for a boundary condition of the third kind (heat
transfer from the environment according to the Newton convection law) on their cooling sur-
face.

We use the following assumptions: a) the thermophysical characteristics of the solidify-
ing alloy (ρ, C, λ, $\mathscr{L}$) are assumed invariant; b) a stabilized mode of ingot cooling is con-
sidered, when the temperature T_c of the cooling medium and the coefficient of heat elimina-
tion on the ingot surface α are assumed invariant; c) the ingot crystallization front is a
spatially distributed two-phase zone within whose limits the content of the solid phase is
determined uniquely by the local temperature and can be reflected by a third degree poly-
nomial [3]: $\psi = \beta_1\Theta + \beta_2\Theta^2 + \beta_3\Theta^3$, where $\Theta = (T_{liq}-T)/(T_{liq}-T_c)$.

If the initial temperature of the ingot exceeds the solidus temperature, then the
solidification process can be separated into three stages: 1) from the beginning of cooling
to the time of achievement of the solidus temperature on the ingot surface; 2) from the end
of stage 1) to the time of achievement by solidification front r = b of ingot axis; 3) from
the end of stage 2) to the time of the reduction of the temperature on the ingot axis to the
solidus temperature (Fig. 2).

The equation of nonstationary heat conduction for the ingot two-phase zone has the form
[3]

$$\rho C \frac{\partial T}{\partial t} = \mathrm{div}\,(\lambda \,\mathrm{grad}\,T) + \rho\mathscr{L}\frac{\partial\psi}{dt}, \tag{1}$$

where the last component on the right side characterizes the intensity of the phase-transi-
tion heat source, which is directly proportional to the local rate of solidification. In par-
ticular, for a temperature field with axial symmetry, (1) takes the form

$$\rho C \frac{\partial T}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(\lambda\frac{\partial T}{\partial r}\right) + \rho\mathscr{L}\frac{\partial\psi}{\partial t}. \tag{2}$$

Let us supplement (2) with initial and boundary conditions

$$t = 0,\; T = T\,; \tag{3}$$
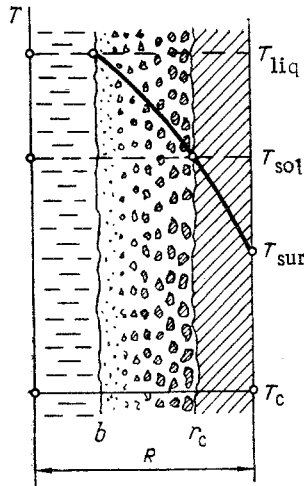
$$r = 0,\; \frac{\partial T}{\partial r} = 0; \tag{4}$$

Fig. 1. Diagram of the location of the two-phase zone bounded by liquidus (b) and solidus (r_c) isotherms in the solidifying ingot section.

$$r = R, \quad -\lambda \frac{\partial T}{\partial r} = \alpha (T_{sur} - T_c). \tag{5}$$

The solution of the problem defined by (2)-(5) is by the method of integral relations according to [4]. We multiply both sides of (2) by rdr and integrate with respect to the coordinate r between the limits 0 and R; we consequently obtain the relationship

$$\frac{dQ}{dt} = -\lambda \left(R \cdot \frac{\partial T}{\partial r} \right)\bigg|_{r=R}, \tag{6}$$

where

$$Q = \rho \mathcal{L} \int_0^R \psi' r dr - \rho C \int_0^R T r dr. \tag{7}$$

Taking account of the stabilized nature of ingot cooling, we take the following temperature distribution over its section:

for the cooling stages 1) and 2)

$$T(r) = T_{liq} - \frac{(T_{liq} - T_c)(R - b)}{R - b + \frac{n\lambda}{\alpha}} \left(\frac{r - b}{R - b} \right)^n \quad \text{for } b \leqslant r \leqslant R,$$

$$T(r) = T_{liq} = \text{const} \quad \text{for } 0 \leqslant r \leqslant b; \tag{8}$$

for the cooling stage 3)

$$T(r) = T_{sol} - \frac{(T_{sol} - T_c)(R - r_c)}{R - r_c + \frac{n\lambda}{\alpha}} \left(\frac{r - r_c}{R - r_c} \right)^n \quad \text{for } r_c \leqslant r \leqslant R,$$

$$T(r) = T_{sol} + (T_{liq} - T_{sol}) \frac{r_c}{r_{c_1}} \left(1 - \frac{r^m}{r_c^m} \right) \quad \text{for } 0 \leqslant r \leqslant r_c, \tag{9}$$

where r_{c1} is the coordinate of the solidus isotherm at the beginning of the ingot cooling stage 3). It is easy to see that (8) and (9) satisfy the boundary condition (5) on the ingot surface being cooled. The values of the exponents n and m are assumed constant for each cooling stage. The numerical solution of the problem (2) and (5), executed by applying a mesh method and an electronic computer, yields a foundation for selecting the exponents n and m equal to 1 ± 0.05 and 3 ± 0.25 respectively. Substituting the coordinate functions (8) and (9) into (7) and then into the integral relation (6), we obtain an ordinary differential equation whose integral connects the liquidus and solidus isotherm coordinates (b, r_c) with the time t. In dimensionless form the solution of the problem has the form:

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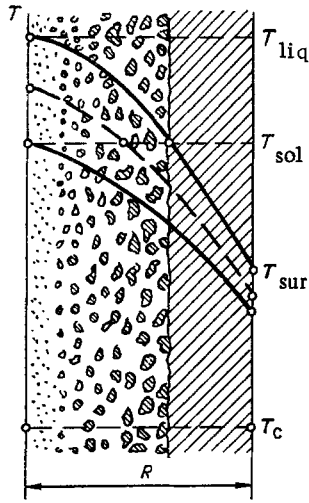Fig. 2. Schematic temperature distribution in the ingot in the third stage of cooling.

for stage 1)

$$\mathrm{Fo} = B_1(1-z) + B_2(1-z^2) + B_3(1-z^3) + B_4\left(1 + \frac{1-z}{N}\right)$$

$$+ B_5\frac{1-z}{1+N-z} + B_6\frac{2(N+1)z-z^2-(1+2N)}{(1+N-z)^2}, \tag{10}$$

where

$$B_1 = \frac{1}{n}\left[(1+K_1\beta_1)\left(\frac{N^2-1}{n+1} + \frac{2+N-N^2}{n+2}\right) + K_1\beta_2\left(\frac{2N^2+N-1}{2n-1} + \right.\right.$$

$$\left.\left. + \frac{2-2N^2}{2n+2}\right) + K_1\beta_3\left(\frac{3N^2+2N-1}{3n+1} - \frac{3N^2+N-2}{3n+2}\right)\right];$$

$$B_2 = \frac{1}{2n}\left[(1+K_1\beta_1)\left(\frac{N+3}{n+1} - \frac{N+4}{n+2}\right) + K_1\beta_2\left(\frac{3}{2n+1} - \right.\right.$$

$$\left.\left. - \frac{4}{2n+2}\right) + K_1\beta_3\left(\frac{3-N}{3n+1} - \frac{4-N}{3n+2}\right)\right];$$

$$B_3 = -\frac{2}{3n}\left[(1+K_1\beta_1)\left(\frac{1}{n+1} - \frac{1}{n+2}\right) + K_1\beta_2\left(\frac{1}{2n+1} - \right.\right.$$

$$\left.\left. - \frac{1}{2n+2}\right) + K_1\beta_3\left(\frac{1}{3n+1} - \frac{1}{3n+2}\right)\right];$$

$$B_4 = -\frac{N^2}{n}\left[(1+K_1\beta_1)\left(\frac{N+1}{n+1} - \frac{N}{n+2}\right) + K_1\beta_2\left(\frac{4N+3}{2n+1} - \right.\right.$$

$$\left.\left. - \frac{4N}{2n+2}\right) + K_1\beta_3\left(\frac{10N+6}{3n+1} - \frac{10N}{3n+2}\right)\right];$$

$$B_5 = 2K_1\frac{N^2}{n}\left[\beta_2\left(\frac{N+1}{2n+1} - \frac{N}{2n+2}\right) + \beta_3\left(\frac{5N+4}{3n+1} - \frac{5N}{3n+2}\right)\right];$$

$$B_6 = \frac{3}{2}\frac{N^2}{n}K_1\beta_3\left(\frac{N+1}{3n+1} - \frac{N}{3n+2}\right);$$

for stage 2)

$$\mathrm{Fo} = C_1(z_0-z) + C_2(z_0^2-z) + C_3(z_0^3-z) + C_4\ln\frac{1+N-z}{1+N-z_0}, \tag{11}$$
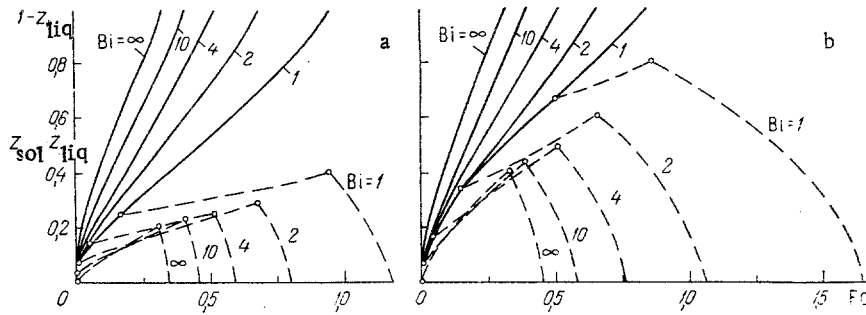
Fig. 3. Progress of the front of the two-phase zone $1 - z_{liq}$ (solid lines) and change in its width $z_{sol} - z_{liq}$ (dashes) during solidification of a cylindrical ingot for the case $K_1 = 1$ and for $\Theta_{sol} = 0.2$ (a) and 0.4 (b).

where

$$C_1 = \frac{1}{n}\left[\frac{2 + N - N^2}{n + 2} + \frac{N^2 - 1}{n + 1} - K_1(N + 1)^2(d_2 - 2d_1)\right];$$

$$C_2 = \frac{1}{2n}\left[K_1(N + 1)(1 + 3d_2 - 4d_1) + \frac{N + 3}{n + 1} - \frac{N + 4}{n + 2}\right];$$

$$C_3 = -\frac{1}{3n}\left[K_1(1 + 2d_2 - 2d_1) + 2\left(\frac{1}{n + 1} - \frac{1}{n + 2}\right)\right];$$

$$C_4 = -\frac{N^2}{n}\left(\frac{N + 1}{n + 1} - \frac{N}{n + 2}\right);$$

$z_0 = b_0/R$ is the initial relative coordinate of the liquidus isotherm for stage 2);

$$d_1 = \Theta_{sol}^2\left(-\frac{1}{2} + \frac{\beta_1\Theta_{sol}}{n + 2} + \frac{\beta_2\Theta_{sol}^2}{2n + 2} + \frac{\beta_3\Theta_{sol}^3}{3n + 2}\right); \qquad d_2 = \Theta_{sol}\left(-1 + \frac{\beta_1\Theta_{sol}}{n + 1} + \frac{\beta_2\Theta_{sol}^2}{2n + 1} + \frac{\beta_3\Theta_{sol}^3}{3n + 1}\right);$$

for stage 3)

$$\text{Fo} = D_1(z_1 - z_c) + D_2(z_1^2 - z_c^2) + D_3(z_1^3 - z_c^3) + D_4(z_1^4 - z_c^4) + D_5(z_1^5 - z_c^5) + D_6(z_1^6 - z_c^6) + D_7\ln\frac{1 + N - z_c}{1 + N - z_1},$$

$$(12)$$

where

$$D_1 = \frac{1}{n}\left(\frac{N^2 - 1}{n + 1} + \frac{2 + N - N^2}{n + 2}\right);$$

$$D_2 = \frac{1}{2n}\left[\frac{N + 3}{n + 1} - \frac{N + 4}{n + 2} - A_1(N + 1)\right];$$

$$D_3 = \frac{1}{3n}\left[2\left(\frac{1}{n + 1} - \frac{1}{n + 2}\right) + A_3(N + 1) - A_1\right];$$

$$D_4 = \frac{1}{4n}[A_2 - A_3(N + 1)]; \quad D_5 = \frac{1}{5n}[A_3 - A_4(N + 1)]; \quad D_6 = \frac{A_4}{6n};$$

$$D_7 = -\frac{N^2}{n}\left(\frac{N + 1}{n + 1} - \frac{N}{n + 2}\right);$$

$$A_1 = \frac{K_1}{1 - \Theta_{sol}}(-1 + \beta_1\Theta_{sol} + \beta_2\Theta_{sol}^2 + \beta_3\Theta_{sol}^3);$$

$$A_2 = \frac{3a_m\Theta_{sol}}{z_1(1 - \Theta_{sol})}[1 + K_1(\beta_1 + 2\beta_2\Theta_{sol} + 3\beta_3\Theta_{sol}^2)];$$

$$A_3 = \frac{4b_mK_1\Theta_{sol}^2}{z_1^2(1 - \Theta_{sol})}(\beta_2 + 3\beta_3\Theta_{sol}); \quad A_4 = \frac{5c_mK_1\beta_3\Theta_{sol}^3}{z_1^3(1 - \Theta_{sol})}; \quad z_c = \frac{r_c}{R};$$

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TABLE 1. Duration of the Separate Stages of
Ingot Solidification Expressed by the Similarity
Number Fo_i (i = 1, 2, 3) and Total Duration of
Solidification $(Fo_\Sigma = Fo_1 + Fo_2 + Fo_3)$ of a
Cylindrical Ingot for Different Values of the
Number Bi for $\beta_1 = \Theta^{-1}$, k = 1

Bi	Θ_{sol}	Fo_1	Fo_2	Fo_3	Fo_Σ
1	0	—	—	—	0,927
	0,2	0,161	0,788	0,222	1,171
	0,4	0,500	0,360	0,783	1,643
2	0	—	—	—	0,646
	0,2	0,041	0,638	0,121	0,800
	0,4	0,146	0,512	0,395	1,053
4	0	—	—	—	0,491
	0,2	0,009	0,503	0,080	0,592
	0,4	0,037	0,473	0,249	0,759
10	0	—	—	—	0,388
	0,2	0,001	0,401	0,060	0,462
	0,4	0,007	0,382	0,178	0,567
∞	0	—	—	—	0,306
	0,2	0	0,317	0,044	0,361
	0,4	0	0,321	0,134	0,455

$z_1 = r_{c_1}/R$ is the relative coordinate of the solidus isotherm at the beginning of stage 3);

$$a_m = \frac{1}{2}\left(\frac{2}{m+2} - 1\right); \quad b_m = \frac{1}{2} - \frac{2}{m+2} + \frac{1}{2m+2};$$

$$c_m = \frac{1}{2} + \frac{3}{m+2} - \frac{3}{2m+2} + \frac{1}{3m+2};$$

$$K_1 = \frac{\mathscr{L}}{C(T_{liq} - T_c)}; \quad N = \frac{n}{Bi}; \quad Bi = \frac{\alpha R}{\lambda}.$$

Selecting the coefficients $\beta_1 - \beta_3$ by different methods, we can reflect the specific form of the phase diagram in the analyses. In the simplest case of parallel liquidus and solidus lines, we take $\beta_1 = 1/\Theta_{sol}$, $\beta_2 = \beta_3 = 0$. The physical meaning of this assumption is that the heat of the phase transition and the enthalpy of the melting are distributed uniformly within the crystallization band T_{liq}-T_{sol}. As an example of utilization of (10)-(12), graphs are represented in Fig. 3 for the progress of the two-phase zone front in a solidifying melt, as is also the change of its relative width in time for a given similarity number $K_1 = 1$; Bi = 1; 2; 4; 10 and ∞; $\Theta_{sol} = 0.2$ and 0.4. As is seen from the figure, the two-phase zone width grows continuously during stages 1) and 2), reaches the maximum at the beginning of stage 3), and then diminishes rapidly to zero. Comparison of the graphs shows that as the alloy crystallization range increases (i.e., complex Θ_{sol}) the total duration of ingot solidification grows (see the table also). A substantial increase in the design duration of the ingot solidification with extraction of the heat of phase transition taken into account in a certain finite temperature range T_{liq}-T_{sol} as compared with the traditional formulation of the Stefan problem (the phase transition occurs by a jump at a certain temperature $T = T_{cr}$) justifies utilization of (10)-(12), which have a sufficiently awkward form. On the other hand, the method elucidated above represents the possibility of determining the two-phase zone width z_{sol}-z_{liq} by a design means, depending on the specific ingot cooling conditions and the thermophysical properties of the alloy. This circumstance is of no little importance since defects (porosity, impurity segregation) originate in the two-phase zone, whose quantity depends, in a governing manner, on the duration of the alloy residence in the two-phase state [5].

NOTATION

T, temperature; T_c, temperature of the cooling medium; T_{liq}, T_{sol}, equilibrium values of the liquidus and solidus temperatures of the solidfying alloy; t, time; r, a coordinate; ρ, C, λ, $\mathscr{L}$, mass density, specific heat, heat conductivity, and melting enthalpy coefficients of the alloy; V_s, V_ℓ, specific volumes of the solid and liquid phases within the control volume V_0 of the two-phase zone $(V_0 = V_s + V_\ell)$; ψ, relative quantity of solid phase; b and r_c, coordin-

ates of the liquidus and solidus isotherms; $z_{liq} = b/R$; $z_{sol} = r_c/R$; $Fo = at/R^2$, $Bi = \alpha R/\lambda$, $K_1 = \mathcal{L}/C(T_{sol} - T_{sol})$, similarity numbers (dimensionless time, cooling criteria, and thermophysical properties of the alloy), $a = \lambda/\rho C$.

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CONFIGURATION OF THERMALLY LOADED COMPONENTS IN ELECTRONIC EQUIPMENT

G. N. Dul'nev and A. O. Sergeev
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Formulation of the Problem. One of the problems of electronic design is that of creating an optimum configuration of components from the thermal viewpoint. By configuration we mean determination of the positions of the modules and components making up a piece of equipment, as well as determination of positions of individual topological elements upon a printed circuit card. Usually component configuration is decided from mounting and connection considerations, but with increased component density and heat liberation ever more attention must be given to thermal criteria. An optimum configuration results in reduction of component temperatures, leading to an increase in reliability of the equipment as a whole. Criteria for evaluating component placements are chosen from increased reliability considerations. Temperature dependences of failure rates have been determined experimentally and can be found in handbooks on reliability [1, 2]. These experimental dependences can be approximated well by exponential functions [3, 4]; however to the accuracy required for practical purposes within a limited temperature range failure intensity can be represented as a linear function of temperature [5]:

$$\lambda(t) = C_1 + C_2 t, \tag{1}$$

where C_1, C_2 are approximation coefficients. Hence the criterion of reducing the net component failure rate leads to a need to reduce the net component temperature:

$$\min \varphi_1 = \min \sum_{j=1}^{n} t_j. \tag{2}$$

In a number of cases it becomes necessary to achieve temperature equalization by reconfiguration of components. This occurs when it is necessary to minimize temperature stresses in a module or decrease electrical imbalance in a circuit caused by differing temperatures of its components etc. [2, 6]. The temperature equalization requirement can be written in the form

$$\min \varphi_2 = \min \sum_{j=1}^{n} (t_j - \overline{t})^2, \quad \overline{t} = \frac{1}{n} \sum_{j=1}^{n} t_j. \tag{3}$$

We will now formulate the configuration problem. The electrical connections between the components, their dimensions, and the heat which they liberate are known. A set of limits is specified for temperature, volume, and cost. It is then necessary to create a configuration which will produce an extremum in the chosen criteria for the specified limitations. Two approaches to this problem are possible. In the first the problem is reduced to arranging the components by some algorithm with the goal of minimizing the chosen thermal regime criteria, for example, in the form of Eq. (2) or (3), with limitations on the volume and location of elements. In the second approach the volume of the equipment is minimized with